Relative homotopy invariants of the type of the Lusternik–Schnirelmann category

Relative Homotopie–Invarianten des Types der Kategorie von Lusternik–Schnirelmann

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Contents

Introduction

Among the numerous homotopy invariants the *category of Lusternik-Schnirelmann*, or *LS-category*, of a topological space has aroused much interest since its definition in 1934 [LS34]. For example it was shown that it is related to another invariant: the *cone-length* of a space [Fox41], [Gan67], [Cor95]. Moreover LS-category can be extended to continuous maps in three different ways [Fox41], [Fad85], thus generating the *F-category*, the *R-category* and the *LS-category* of a map, which are analogous to the *sectional category* of a fibration [Sch66]. Finally Félix and Halperin [FH83] gave a new dimension to the LS-category by transferring it into the context of rational homotopy theory: they gave a method to compute its rationalization directly in the category of *commutative cochain algebras* (in short: cca's). They also rationalized the F-category of a map.

In this thesis we are particularly interested in relative invariants of the type of the LS-category, such as F-category, R-category, LS-category, sectional category and cone-length of a map. In chapter 1 we introduce a few tools which are very useful to define the various relative categories: *homotopy push-outs*, *homotopy pull-backs* and *joins*. Then we give a brief description of rational homotopy theory in chapter 2: we state the equivalence of categories underlying it which links topological spaces and commutative cochain algebras (in short: cca's). We also define (relative) *Sullivan algebras*, which are particularly nice to deal with, and can be used as building blocks when modelizing some topological constructions such as joins.

Chapter 3 is devoted on the one hand to a description of the original LS-category and cone-length. In particular we give three equivalent definitions of the LS-category: in terms of coverings, of *fat wedges* and of *Ganea maps*, constructed by taking consecutive joins. We also give bounds for the LS-category and the cone-length of a product of spaces. On the other hand we introduce the F-category, the R-category and the LS-category of maps, giving for each of them three equivalent definitions, as well as the *cone-length of a map* [Mar98].

In chapter 4 we find a bound for the cone-length of a product of maps and use it to obtain bounds for the F-category, the R-category and the LS-category of a product of maps.

Chapter 5 contains a summary of part of Félix and Halperin's paper [FH83] giving a rationalization of the absolute LS-category and of the F-category and their characterization directly in the rational context. We then introduce a rationalization of the R-category and the relative LS-category and we state our main theorem, allowing to compute them directly in the cca setting: we use any Sullivan model of the morphism f to construct new morphisms π_m with target space $\mathfrak{F}_m$, $m \geq 0$. The category of f depends then on the existence of a homotopy retract for some π_m. We give a proof of this assertion in chapter 6 by defining *Ganea algebras* $\mathfrak{G}_m$ and *Ganea morphisms* $\mathfrak{g}_m$, $m \geq 0$ modelling Ganea spaces and maps, and then by building (homotopy) commutative diagrams involving π_m and $\mathfrak{g}_m$, $m \geq 0$, which relate the existence of a homotopy retract for π_m to the existence of a homotopy retract for $\mathfrak{g}_m$.

Some applications of the main theorem are given in chapter 7: we show that the R-category can take up any value, and we simplify our main result in case the map being considered is the inclusion of a fibre. Moreover we prove that the rational relative category of a spherical fibration does not depend only on the order of its Euler class as it is the case for its rational sectional category.

Finally we devote our last chapter to the study of a new homotopy invariant: the *sectional category of a sequence of maps*, which generalizes both the sectional category of a fibration and the R-category. In this case as for the classical LS-category we give three equivalent definitions in terms of coverings, of generalized fat wedges and of generalized Ganea spaces. Moreover we rationalize the new invariant and prove a theorem allowing its direct computation in the rational setting.